Geometric k-essence from nonmetricity and late-time cosmology

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SISSA APP seminar, June 2025

Outline

Part I **Introduction**

Why & how to modify gravity

Motivation

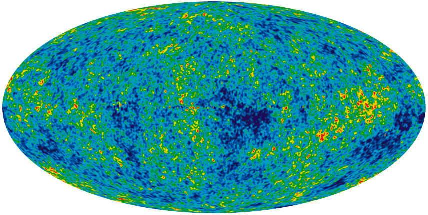
General Relativity appears incomplete in some way

Theoretical challanges:

- Singularities
- Quantum gravity
- Cosmological constant problem

Observational challenges to ΛCDM:

- Origins of dark matter and dark energy
- Observational tensions (e.g., H_0, $\not{S}_8$, Ω_k CMB anomoloies, etc.)



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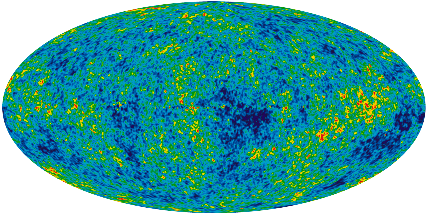

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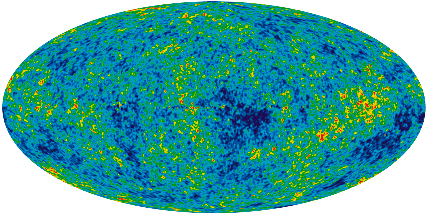
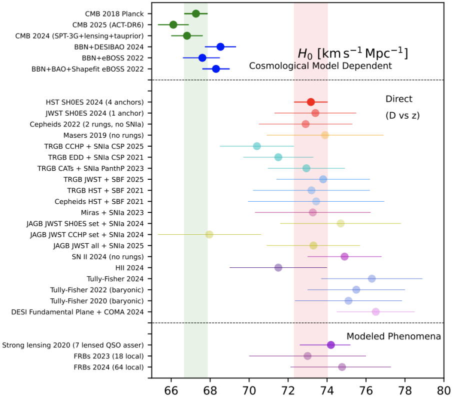


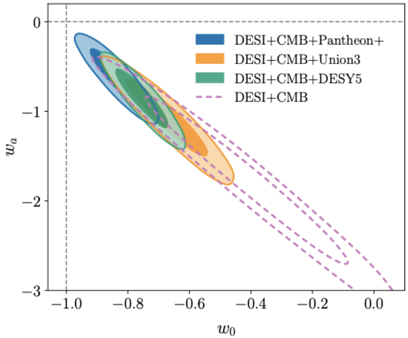
Cosmic tensions

H_0 tension persists



(a) Cosmoverse white paper 2504.01669

Evidence of dynamical dark energy



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2.3σ tension from BAO vs Planck CMB

$2.8 - 4.2\sigma$ evidence for evolving DE (including SN-Ia datasets)

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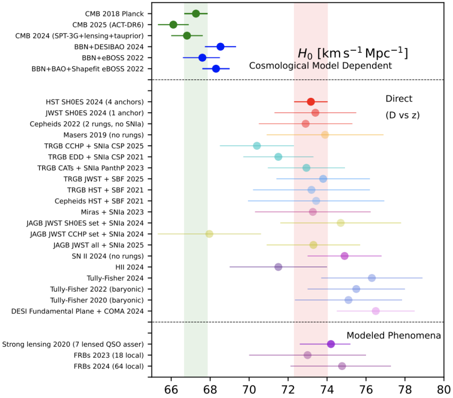

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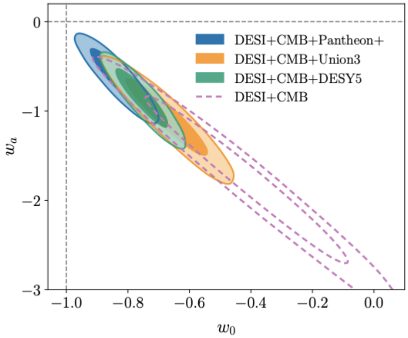

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How do we modify GR?

Lovelock's theorem

The only second-order, divergence-free, symmetric equations constructed solely from the metric and derivable from a diffeomorphism & locally Lorentz invariant action in 4D are

$$G_{\mu\nu} + \Lambda g_{\mu\nu}$$

Modifying GR means breaking some of these conditions.

Beyond General Relativity

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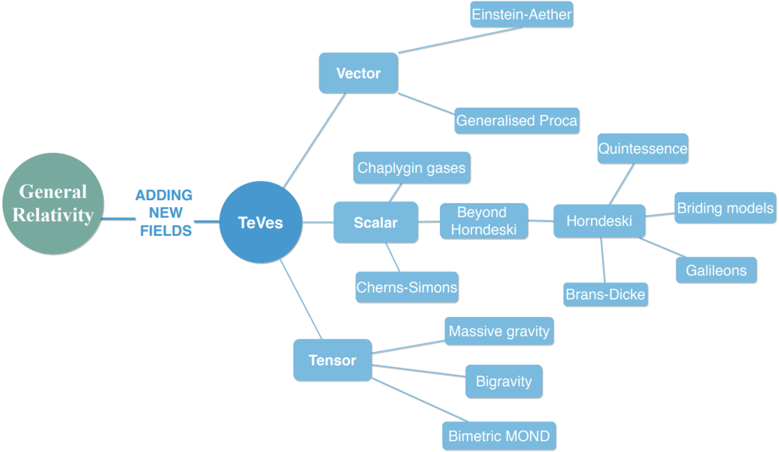
Beyond General Relativity

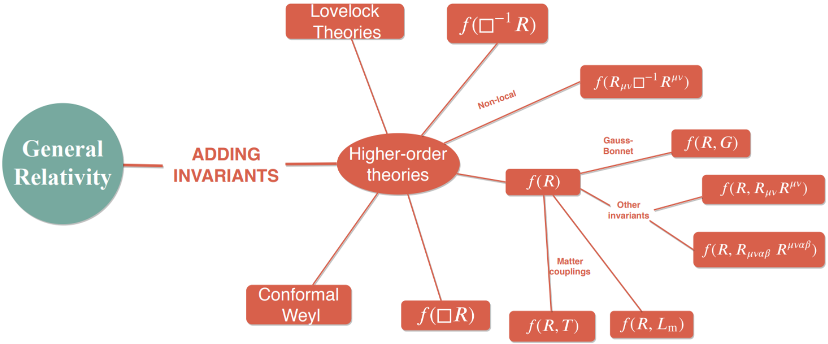
Many popular approaches to modifying GR: **fields**



Source: CANTATA network [2105.12582 gr-qc]

Beyond General Relativity

Many popular approaches to modifying GR: fields, **invariants**



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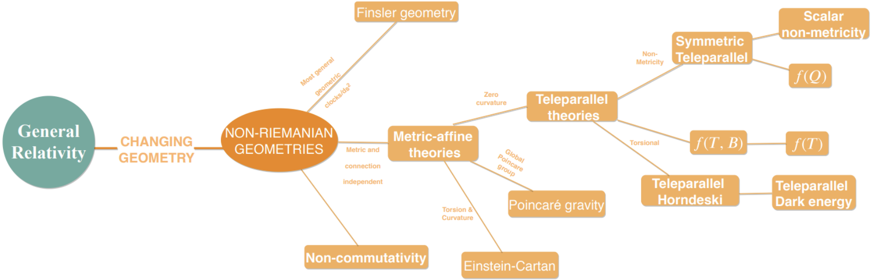
Many popular approaches to modifying GR: fields, invariants & **geometric**



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Modified gravity

All modifications include new degrees of freedom:

Higher invariants

$f(R)$ gravity, quadratic gravity, Gauss-Bonnet gravity, Lovelock gravity, Weyl gravity, conformal gravity, $f(Riemann)$ gravity, infinity derivative gravity, $f(R,T)$ gravity, ...

New fields

Scalar, vector, tensor, massive gravity, Hordenski, Brans-Dicke, Bigravity, non-minimal couplings, ghost condensates, multi-metric gravity, Dilaton, ...

Non-Riemannian geometry

Metric-affine, Teleparallel gravity Einstein-Cartan, Weyl gravity, Poincare gravity, non-commutativity geometry, Finsler geometry, semi-symmetric affine gravity, Yano-Schrodinger gravity, ...

with many equivalences!

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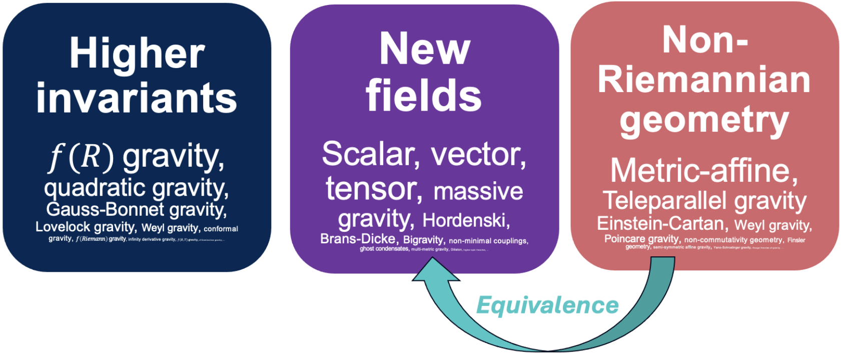


k-essence $\Longleftrightarrow$ nonmetricity

I will focus on the duality between scalar-tensor models & non-Riemannian geometry

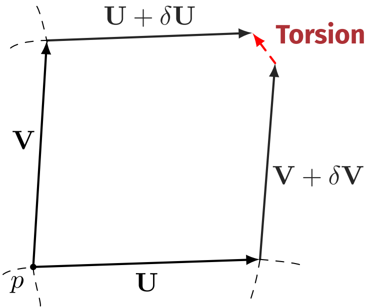
Part II **Mathematical formulation**

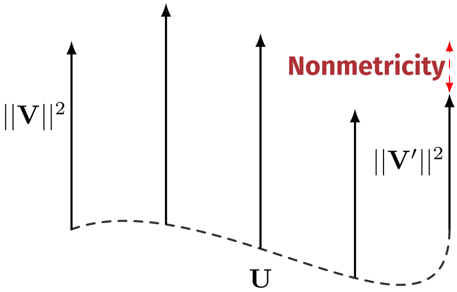
Outline of the geometric framework, action principles with nonmetricity, and the k-essence duality

Geometry

- In general metric-affine space $\{\mathcal{M}, g, \Gamma\}$ we have:

Curvature $R_{\mu\nu\rho}{}^{\gamma}$, Torsion $T_{\mu\nu}{}^{\lambda} = 2\Gamma^{\lambda}_{[\mu\nu]}$, Nonmetricity $Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu}$





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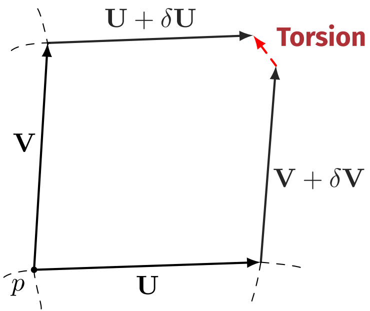

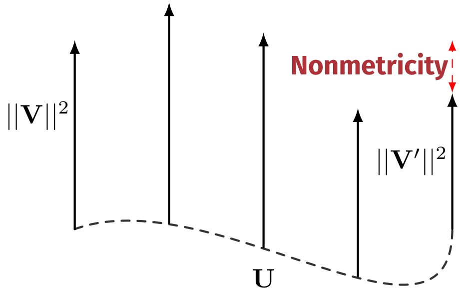

- Post-Riemannian decomposition $\Gamma = \mathring{\Gamma} + N$ with $N \propto T + Q$ leads to

$$R = \underbrace{\mathring{R}}^{\text{Levi-Civita}} + \overbrace{N^{\mu}{}_{\mu\beta}N^{\beta\nu}{}_{\nu} - N^{\mu}{}_{\nu\beta}N^{\beta}{}_{\mu}{}^{\nu}}^{\text{Torsion + Nonmetricity}} + \overbrace{\mathring{\nabla}_{\mu}N^{\mu\nu}{}_{\nu} - \mathring{\nabla}_{\nu}N^{\mu}{}_{\mu}{}^{\nu}}^{\text{Boundaries}}$$

Vectorial nonmetricity

We work in a space $\{\mathcal{M}, g, \Gamma\}$ with **curvature** & a specific form of **vectorial nonmetricity**

Generalised vectorial nonmetricity

$$Q_{\mu\nu\rho} = c_1 \pi_\mu g_{\nu\rho} + c_2(\pi_\rho g_{\mu\nu} + \pi_\nu g_{\mu\rho}) + 2c_3 \pi_\mu \pi_\nu \pi_\rho$$

where π^μ is a vector, c_1, c_2, c_3 are constant parameters

This extends many historical works in the literature:

- Weyl geometric framework: c_1 associated with scale invariance (Weyl, 1918)
- Generalized Weyl vector nonmetricity: adds c_2 terms (Aringazin & Mikhailov, 1991)
- Linear vectorial distortion: include c_1 and c_2 plus linear torsion terms (Jiménez and Koivsto, 2016)

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- On FLRW backgrounds most general type of nonmetricity (Iosifidis 2003.07384) takes the form

$$Q_{\mu\nu\rho} = A(t) u_\mu h_{\nu\rho} + B(t)(u_\rho h_{\mu\nu} + u_\nu h_{\mu\rho}) + 2C(t) u_\mu u_\nu u_\rho$$

  where A, B, and C are aribtrary functions of time.
- Reduces exactly to our form in the simple case where $A \propto B \propto C$

Further motivation:

- 'Natural' extension (all allowed terms without contractions $\pi_\mu \pi^\mu$)
- Phenomenological reasons: the new c_3 terms play a key role in describing dark energy in cosmology

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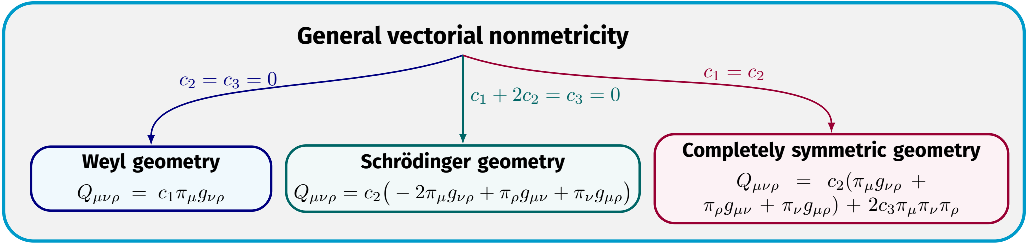

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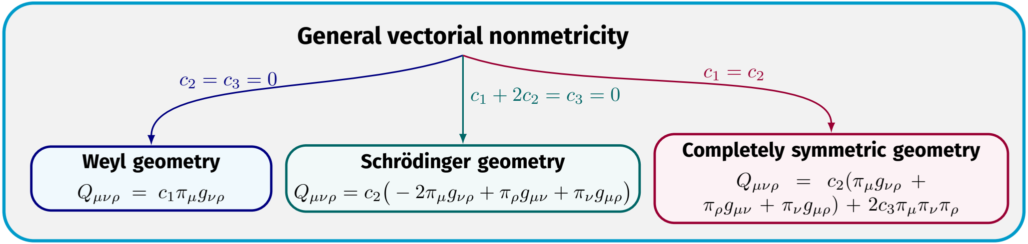

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Gravitational actions

Construct our model using action principles:

- Simplest linear action $\mathcal{L} = R$ using post-Riemannian decomposition

$$R = \mathring{R} + \left(3c_1c_2 + 3c_2^2 - \frac{3c_1^2}{2}\right)\pi_\mu\pi^\mu + 3c_2c_3\pi_\mu\pi^\mu\pi_\lambda\pi^\lambda + 3(c_2 - c_1)\mathring{\nabla}_\mu\pi^\mu$$

- Implement integrability via Lagrange multipliers

$$\mathcal{L}_\lambda = \lambda^\mu\left(\pi_\mu - \mathring{\nabla}_\mu\phi\right)$$

- Full gravitational action

$$S[g,\pi,\phi,\lambda,\psi] = \frac{1}{2\kappa}\int\sqrt{-g}Rd^4x + S_\lambda[g,\pi,\phi,\lambda] + S_{\mathrm{m}}[g,\psi]$$

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- Variations in $(g, \pi, \phi, \lambda, \psi)$ lead to field equations:

$$\mathring{G}_{\mu\nu} + \mathring{\nabla}_\mu\phi\mathring{\nabla}_\nu\phi\Big(b_1 + b_2\mathring{\nabla}_\lambda\phi\mathring{\nabla}^\lambda\phi\Big) - \frac{1}{2}g_{\mu\nu}\mathring{\nabla}_\lambda\phi\mathring{\nabla}^\lambda\phi\Big(b_1 + \frac{b_2}{2}\mathring{\nabla}_\rho\phi\mathring{\nabla}^\rho\phi\Big) = \kappa T_{\mu\nu}$$

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with two parameters

$$b_1 = 3c_1c_2 + 3c_2^2 - \frac{3c_1^2}{2} \quad , \quad b_2 = 6c_3c_2$$

- *On-shell* reveals direct equivalence with *purely kinetic* k-essence model

$$P(X) = -b_1X + b_2X^2 \qquad \text{where} \;\; X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$$

where $b_1 = -1$ and $b_2 = 0$ is *canonical* $\mathcal{L}_\phi = -\nabla_\mu\phi\nabla^\mu\phi/2$

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k-essence action

Equivalent to the purely kinetic k-essence model

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where b_1 and b_2 are determined by nonmetricity parameters c_i

- **Weyl geometry** $c_2 = c_3 = 0$

$$b_1 = -\frac{3c_1^2}{2} \quad b_2 = 0$$

- **Schrödinger geometry** $c_1 = -2c_2$, $c_3 = 0$

$$b_1 = -9c_2^2 \quad b_2 = 0$$

- **Completely symmetric geometry** $c_1 = c_2$

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k-essence stability

What can we say about the stability of the k-essence models?

$$P(X) = -b_1 X + b_2 X^2$$

Introduce the energy density ρ_ϕ and adiabatic sound speed squared c_s^2

$$\rho_\phi := 2XP_{,X} - P = -X(b_1 - 3b_2 X)$$
$$c_s^2 := \frac{P_{,X}}{\rho_{\phi,X}} = \frac{b_1 - 2b_2 X}{b_1 - 6b_2 X}$$

To avoid both classical and quantum instabilities we can require these to be non-negative

$$\rho_\phi \geq 0 \;\&\; c_s^2 \geq 0 \implies b_1 \leq 0 \;\&\; b_2 \geq 0$$

A more detailed approach will reveal stable regions satisfying $\rho_\phi \geq 0$, $c_s^2 \geq 0$ but with $b_1 \not\leq 0$, $b_2 \not\geq 0$ – which is non-trivial

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Part III **Cosmological dynamical systems**

Qualitative studies of the cosmological model and a detailed look at stability

Dynamical systems theory introduction

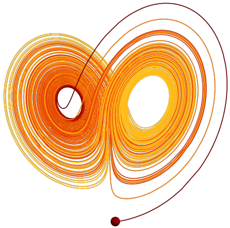
Overview of dynamical systems theory

- Used to study non-linear differential equations which may not be exactly solvable
- Describes the behaviour and time-evolution of dynamical systems
- Emphasis on qualitative over quantitative behaviour
- Can use linear stability theory around equilibrium points (*local theory - Hartman–Grobman theorem*)
- Many advanced geometric methods available

Dynamical systems theory introduction

ODE's of the form

$$\dot{x}_1 = f_1(x_1, ..., x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, ..., x_n)$$



where $\mathbf{x} = (x_1, ..., x_n) \in \mathcal{P} \subseteq \mathbb{R}^n$ and $\dot{x}$ denotes the derivative w.r.t some variable $t \in \mathbb{R}$

- $\mathcal{P}$ is the *phase space*
- The system is *autonomous* if there is no explicit dependence on t
- The map $\mathbf{f} : \mathcal{P} \to \mathcal{P}$ is a vector field on $\mathbb{R}^n$
- Solutions $\psi(t)$ are trajectories along the flow of $\mathbf{f}$, known as *orbits*

Dynamical systems theory introduction

Fixed Point

Fixed point $\mathbf{x}_*$ satisfies $\mathbf{f}(\mathbf{x}_*) = 0$

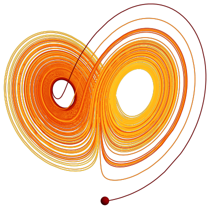
Linearise around fixed point $\mathbf{x}_*$

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_*) + (\mathbf{x} - \mathbf{x}_*) \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_*} + \ \ldots$$

Local evolution determined by *stability matrix*

$$J|_{\mathbf{x}=\mathbf{x}_*} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_*} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_*}$$

Eigenvalues λ determine stability



Hyperbolic Point

Fixed point is hyperbolic iff $\mathrm{Re}(\lambda_\mathrm{i}) \neq 0$

Dynamical systems theory introduction

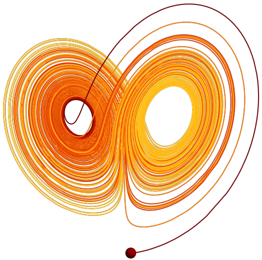
Linear stability classification:

Fixed point is **stable** if $\mathrm{Re}(\lambda_\mathrm{i}) < 0$

Fixed point is **unstable** if $\mathrm{Re}(\lambda_\mathrm{i}) > 0$

Fixed point is a **saddle** if at least two $\mathrm{Re}(\lambda_\mathrm{i})$ have opposite signs

More detailed classifications including $\mathrm{Im}(\lambda)$*, e.g., spirals, centers, chaos, etc.*



Cosmological dynamical systems

Work on flat FLRW background ($\pi_\mu = \dot{\phi} = \sqrt{2X}$)

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

with a perfect fluid energy-momentum tensor

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + g_{\mu\nu} p$$

The cosmological field equations read

$$3H^2 = \kappa\rho - \frac{1}{2} b_1 \dot{\phi}^2 + \frac{3}{4} b_2 \dot{\phi}^4$$
$$3H^2 + 2\dot{H} = -\kappa p + \frac{1}{2} b_1 \dot{\phi}^2 - \frac{1}{4} b_2 \dot{\phi}^4$$
$$b_1 \Box\phi - 3 b_2 \dot{\phi}^2 \left(\Box\phi + 2H\dot{\phi}\right) = 0$$

and the usual matter continuity equation follows $\mathring{\nabla}_\mu T^\mu{}_\nu = 0$

$$\dot{\rho} + 3H(\rho + p) = 0$$

Cosmological dynamical systems

Perform dynamical systems analysis with linear matter EoS $p = w\rho$

Dynamical variables & Friedmann constraint

$$\Omega_m = \frac{\kappa\rho}{3H^2} \quad , \quad x = -\frac{b_1 X}{3H^2} \quad , \quad y = \frac{b_2 X^2}{3H^2}$$

$$\Omega_m + x + y = 1$$

Differentiate w.r.t $N = \log a$ and use acceleration, scalar field & continuity equation:

Evolution equations

$$\frac{dx}{dN} = x\left(3x + y + 1 + 3w(1 - x - y) - \frac{4x}{x + 2y}\right)$$

$$\frac{dy}{dN} = y\left(3x + y - 1 + 3w(1 - x - y) - \frac{8x}{x + 2y}\right)$$

which we easily* solve to find fixed points

**subtleties on divergent $x = -2y$ line*

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subtleties on divergent $x = -2y$ line

Cosmological dynamical systems

Utilise the k-essence parameters related to (perturbative/quantum) stability

$$\Omega_\phi = x + y \qquad c_s^2 = \frac{1}{3} + \frac{2x}{3(x+2y)}$$

and the k-essence and effective equation of state parameters

$$w_\phi = \frac{3x+y}{3(x+y)} \qquad w_{\text{eff}} = w_\phi \Omega_\phi + w\Omega_m$$

Point	Ω_m	x	y	w_{eff}	c_s^2	stability	conditions	interpretation
A	0	0	1	$1/3$	$1/3$	unstable	$b_2 > 0$	tracking (radiation)
B	0	1	0	1	1	saddle	$b_1 < 0$	kinetic (stiff matter)
C	0	-2	3	-1	0	stable	$b_1 > 0$ & $b_2 > 0$	de Sitter
O	1	0	0	w	–	–	none	matter

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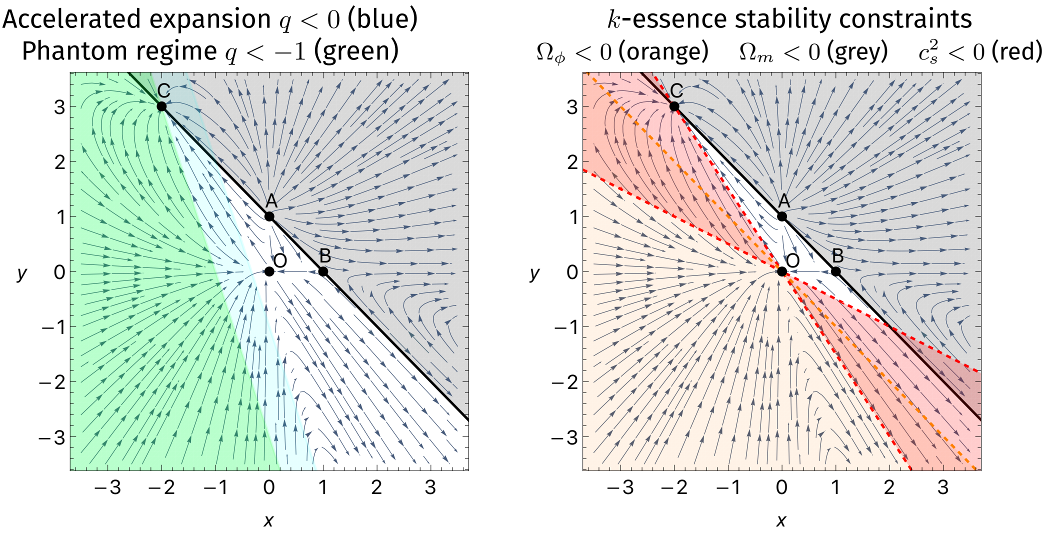
Cosmological dynamical systems



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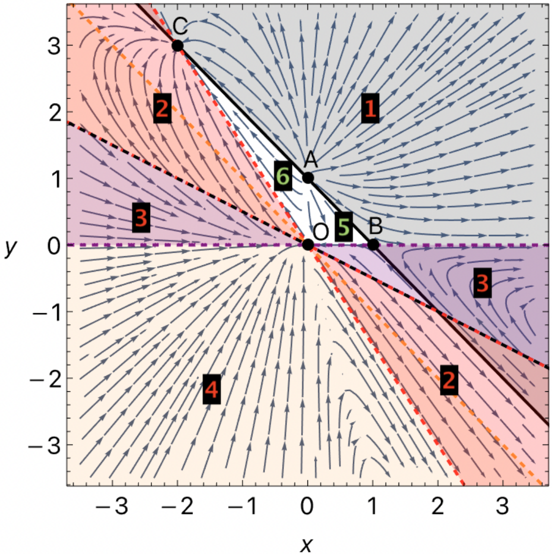


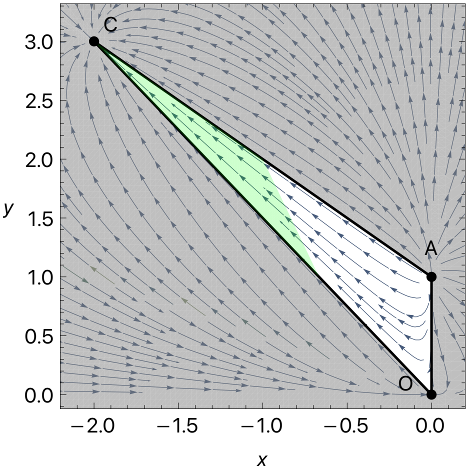
Which regions are theroetically allowed?

1. No - $\Omega_m < 0$
2. No - $c_s^2 < 0$ (and diverges in past)
3. No - $c_s^2 > 1$ (and diverges in past)
4. Probably not - $\Omega_\phi < 0$
5. Yes - but no accelerating solutions
6. Yes

Black dashed line $y = -x/2$ is divergent

Cosmological dynamical systems

Imposing constraints to find (perturbatively) stable & physically viable phase space



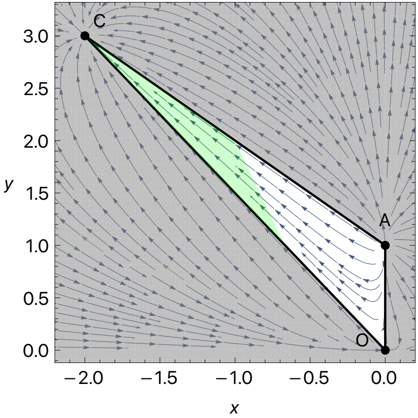
Stable region found by imposing:

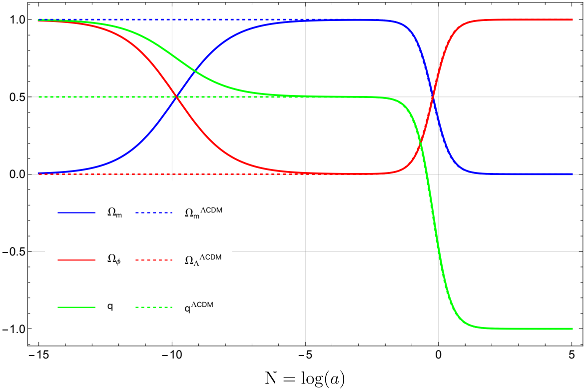
- Energy conditions $0 \leq \Omega_m \leq 1$ and $c_s^2 \geq 0$
- Negative values of x (positive b_1) for existence of de Sitter points

Conclusion: trajectories inside this region remain bounded and pathology-free for all time!

Cosmological dynamical systems

Viable phase space leads to realistic cosmological evolutions





(a) Heteroclinic evolution (A $\to$ O $\to$ C) vs ΛCDM

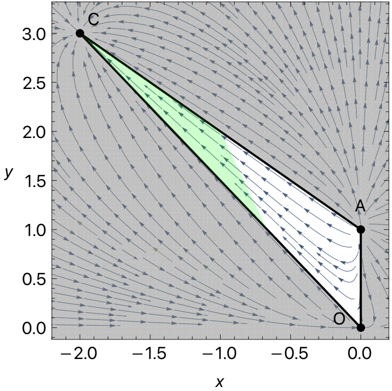
Part IV **Observational constraints**

Testing and constraining the model with real data

Observational constraints - setup

Introduce dimensionless variables related to $\dot{\phi}$ and the parameters $b_1 > 0$, $b_2 > 0$

$$\Phi := \frac{\sqrt{b_1}\dot{\phi}}{H_0} \qquad B := \frac{3b_2 H_0^2}{4b_1^2}$$



- Green region: $c_s^2 \geq 0$, $\Omega_\phi \geq 0$, $q < 0$
- Translating geometric constraints (green region) into new variables:

$$0 \leq \Omega_{m0} < \frac{2}{3}$$

$$\frac{1}{16 - 16\Omega_{m0}} \leq B < \frac{4 - 3\Omega_{m0}}{6(\Omega_{m0} - 2)^2}$$

- We will fix Ω_{m0} to a range of values – explained later!

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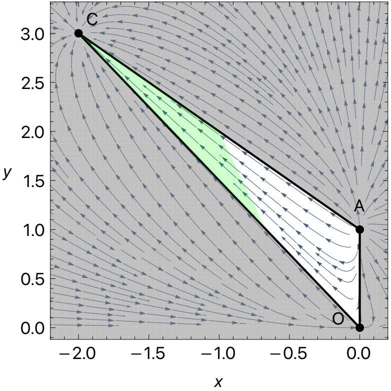

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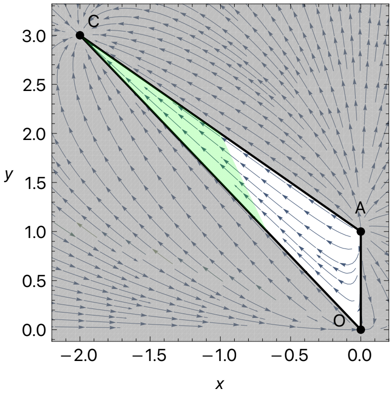

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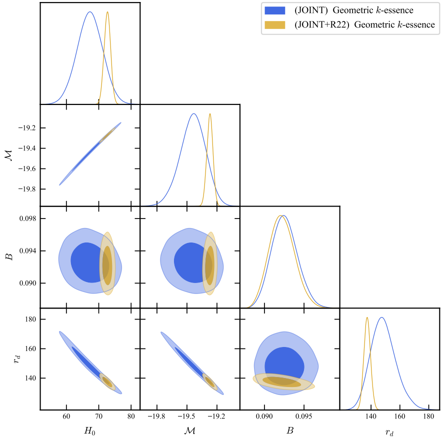
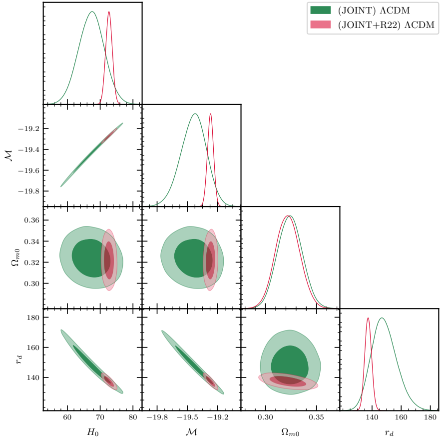
Observational constraints

Constrain model parameters $\{H_0, B, r_d, \mathcal{M}\}$ with late-time data: **cosmic chronometers, Pantheon$^+$ Type Ia supernovae**, and **DESI baryon acoustic oscillations** with *informed priors!*





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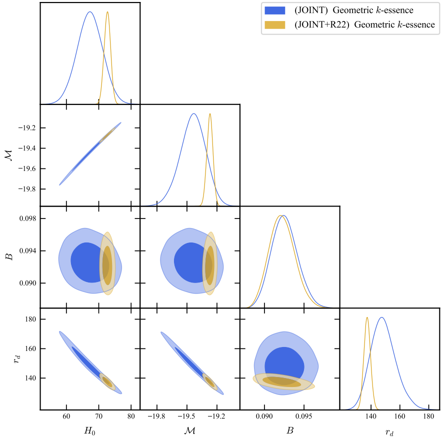

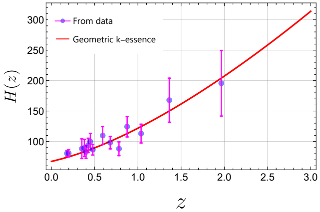

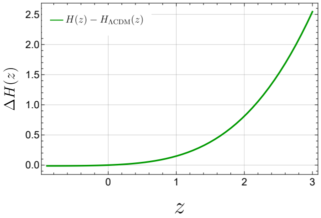

Observational results

Statistically indistinguishable from ΛCDM via late-time observations

Model	Parameter	Prior	JOINT	JOINT + R22
ΛCDM	H_0	$[50, 100]$	67.3 ± 3.9	72.7 ± 1.0
	Ω_{m0}	$[0, 1]$	0.324 ± 0.012	0.322 ± 0.012
	$\mathcal{M}$	$[-20, -18]$	$-19.44^{+0.13}_{-0.12}$	-19.271 ± 0.030
	r_d	$[100, 300]$	$148.6^{+7.5}_{-9.5}$	137.4 ± 2.3
k-essence	H_0	$[50, 100]$	67.3 ± 3.9	72.7 ± 1.0
	B	$[0.0880, 0.1784]$	$0.0925^{+0.0015}_{-0.0018}$	$0.0922^{+0.0015}_{-0.0017}$
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Note B can be related to Ω_{m0} in the ΛCDM model by $\Omega_{m0} = 1 - 1/(16B)$

Model	$\chi^2_{\text{tot,min}}$	P_{tot}	N_{tot}	χ^2_{red}	AIC	ΔAIC	BIC	ΔBIC
ΛCDM	1779.39	4	1728	1.032	1787.39	0	1809.21	0
k-essence	1779.36	4	1728	1.032	1787.36	-0.03	1809.18	-0.03

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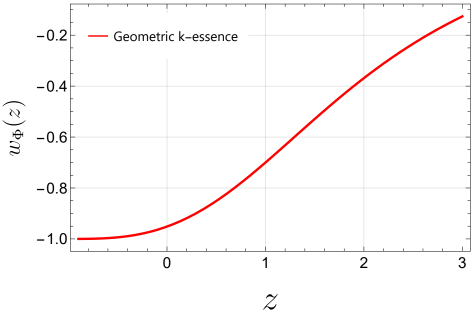
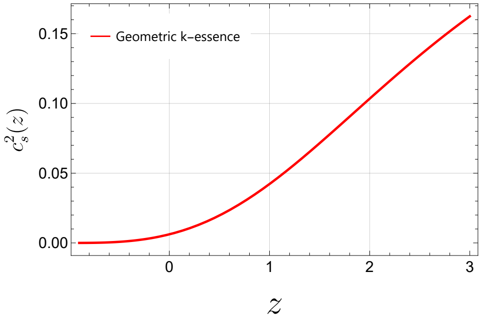
Observational results

But dark energy *is* dynamical!





Late-time limit $(z \to -1)$

$$\Omega_\Phi = 1$$

$$w_\Phi = -1 \quad , \quad c_s^2 = 0$$

Intermediate times

$$0 < \Omega_\Phi < 1$$

$$-1 < w_\Phi < \frac{1}{3} \quad , \quad 0 < c_s^2 < \frac{1}{3}$$

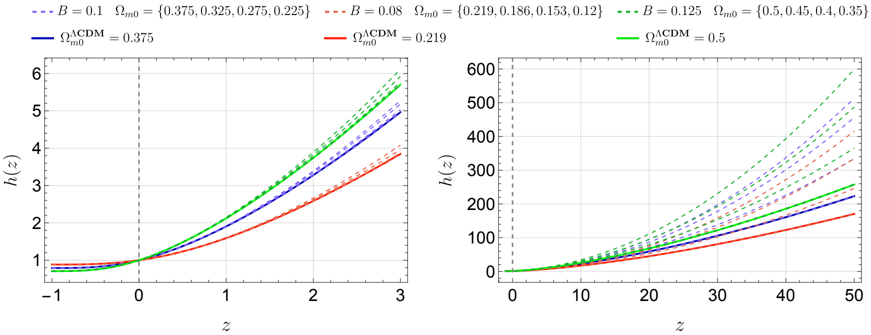
Early-time limit $(z \to \infty)$

$$\Omega_\Phi = 1$$

$$w_\Phi = \frac{1}{3} \quad , \quad c_s^2 = \frac{1}{3}$$

Matter degeneracy

- Ω_{m0} has insignificant effect on low-z evolution



- k-essence field behaves like cosmological constant + matter contributions at late times (Scherrer 04')
- Full analytic solution and small z approximation confirm this behaviour

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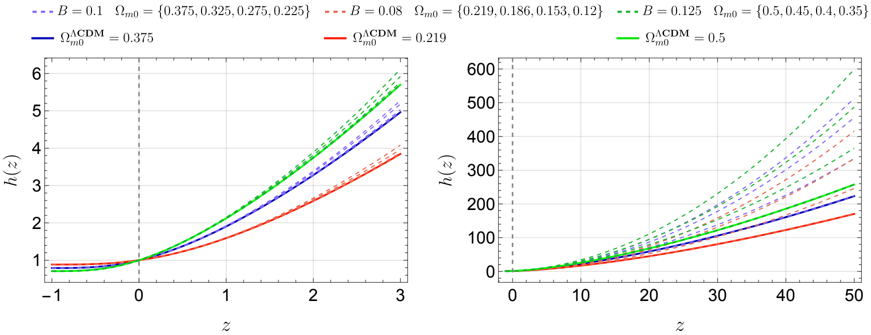

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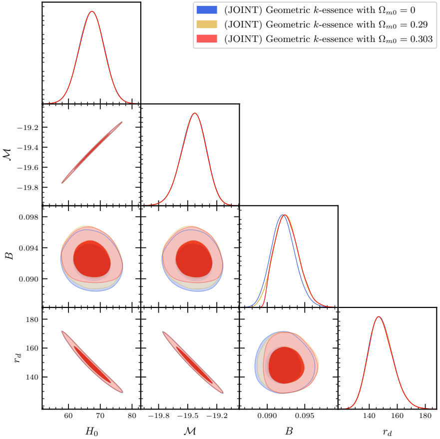
Results completely independent from initial matter density parameter Ω_{m0}



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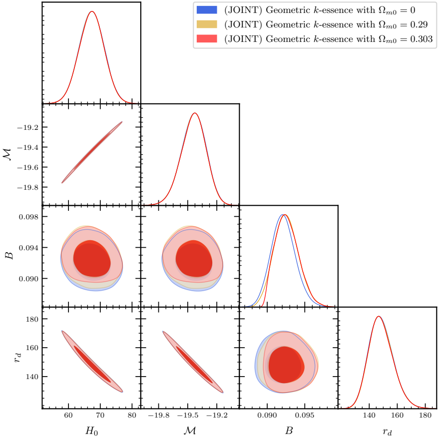

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Matter degeneracy

Optimised results for range of initial matter density parameter Ω_{m0}

k-essence model	Parameter	Prior	JOINT	$\chi^2_{\rm tot,min}$	AIC	BIC
$\Omega_{m_0} = 0$	H_0	$[50, 100]$	67.3 ± 3.9	1778.70	1786.70	1808.52
	$\mathcal{M}$	$[-20, -18]$	-19.44 ± 0.13			
	r_d	$[100, 300]$	$148.5^{+7.5}_{-9.6}$			
	B	$[0.0625, 0.1666]$	$0.0921^{+0.0015}_{-0.0017}$			
$\Omega_{m_0} = 0.1$	H_0	$[50, 100]$	67.3 ± 3.9	1779.03	1787.03	1808.85
	$\mathcal{M}$	$[-20, -18]$	$-19.44^{+0.13}_{-0.12}$			
	r_d	$[100, 300]$	$148.5^{+7.6}_{-9.4}$			
	B	$[0.0694, 0.1708]$	$0.0923^{+0.0015}_{-0.0018}$			
$\Omega_{m_0} = 0.2$	H_0	$[50, 100]$	67.3 ± 4.0	1779.26	1787.26	1809.08
	$\mathcal{M}$	$[20, -18]$	$-19.44^{+0.13}_{-0.12}$			
	r_d	$[100, 300]$	$148.5^{+7.7}_{-9.6}$			
	B	$[0.0781, 0.1748]$	$0.0925^{+0.0015}_{-0.0017}$			
$\Omega_{m_0} = 0.29$	H_0	$[50, 100]$	67.3 ± 3.9	1779.36	1787.36	1809.18
	$\mathcal{M}$	$[-20, -18]$	$-19.44^{+0.13}_{-0.12}$			
	r_d	$[100, 300]$	$148.6^{+7.5}_{-9.5}$			
	B	$[0.0880, 0.1784]$	$0.0925^{+0.0015}_{-0.0018}$			
$\Omega_{m_0} = 0.303$	H_0	$[50, 100]$	67.3 ± 3.9	1779.37	1787.37	1809.19
	$\mathcal{M}$	$[-20, -18]$	$-19.44^{+0.13}_{-0.12}$			
	r_d	$[100, 300]$	$148.5^{+7.6}_{-9.6}$			
	B	$[0.0896, 0.1788]$	$0.0927^{+0.0013}_{-0.0018}$			

Outlook

Summary:

- Geometric motivations and well-defined action principles
- Equivalence with well-known k-essence models
- Surprisingly, never been constrained before:
  due to phase space instabilities & parameter degeneracy
- Dynamical systems analysis used to inform priors
- Shown to be observational indistinguishable from ΛCDM with late-time data
- Many follow-up directions (early-time data, general $P(X)$, nonminimal coupling & non-conservation, higher-order, sources)

Arxiv: 2505.15975 gr-qc
Jointly with Lehel Csillag

Thanks for listening!

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Arxiv: 2505.15975 gr-qc
Jointly with Lehel Csillag

Thanks for listening!

Outlook

Summary:

- Geometric motivations and well-defined action principles
- Equivalence with well-known k-essence models
- Surprisingly, never been constrained before:
  due to phase space instabilities & parameter degeneracy
- Dynamical systems analysis used to inform priors
- Shown to be observational indistinguishable from ΛCDM with late-time data
- Many follow-up directions (early-time data, general $P(X)$, nonminimal coupling & non-conservation, higher-order, sources)

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Thanks for listening!

Appendix

Appendix material

Appendix

Data description:

- **Cosmic Chronometers:** 15 Hubble measurement points extracted based on the differential age method, available together with their covariance matrix
- **Type Ia supernovae**: Pantheon$^+$dataset, which includes 1701 light curves from 1550 Type Ia supernovae (SNe Ia), spanning the redshift range $0.001 \leq z \leq 2.26$
- **Baryon Acoustic Oscillations**: Compilation of BAO data from the DESI DR1 release, which includes multiple galaxy and quasar samples. The data span an effective redshift range from $z_{eff} = 0.295$ to $z_{eff} = 2.33$ with associated measurements of D_M/r_d, D_H/r_d and D_V/r_d, and their uncertainties. r_d is the comoving sound horizon at the drag epoch, which we vary as a free parameter.

Total likelihood:

$$\log \mathcal{L}_{\rm tot} = -\frac{1}{2}\chi^2_{\rm tot} + \rm const$$

and total chi-squared

$$\chi^2_{\rm tot} = \chi^2_{\rm CC} + \chi^2_{\rm SNeIa} + \chi^2_{\rm BAO}$$

with all likelihoods derived from $H(z)$

Appendix

Overview of DESI BAO

- Spectroscopic redshifts $\Rightarrow$ 3D positions of galaxies/quasars.
- Compute clustering (2-point correlation / power spectrum).
- Identify $\sim$150 Mpc BAO scale $\Rightarrow$ standard ruler.
- Extract $H(z)$ (radial) and $D_M(z)$ (transverse), scaled by r_d.
- Enables precise cosmological distance measurements.

Spectroscopic Observations

↓

Redshift Catalog

↓

3D Clustering Map

↓

Measure BAO Feature ($\sim$150 Mpc)

↓

Extract $H(z)$, $D_M(z)$

↓

Compare to r_d $\Rightarrow$ Cosmology

LRG	Luminous Red Galaxy	$0.4 \lesssim z \lesssim 0.8$
ELG	Emission Line Galaxy	$0.6 \lesssim z \lesssim 1.6$
QSO	Quasar Clustering	$0.9 \lesssim z \lesssim 2.1$
Lyman-α	Quasar Absorption	$2.0 \lesssim z \lesssim 3.5$

Full analytic solution

$$\widetilde{X} - \frac{16}{3}B\widetilde{X}^2 + \frac{64}{9}B^2\widetilde{X}^3 - k_0(1+z)^6 = 0\,. \tag{D3}$$

where we have used that $a = 1/(1+z)$. Given that both B and $\widetilde{X}$ must be positive, we also require $k_0 \geq 0$ for real solutions. Only one branch of solution is well-defined for all $z > -1$, which can be written as

$$\widetilde{X}(z) = \frac{1}{4B} + \frac{1+c_z^2}{8Bc_z}\,, \quad \text{with} \quad c_z := \Big(-1 + 6\sqrt{2}\sqrt{Bk_0(18Bk_0(1+z)^6 - 1)}(1+z)^3 + 36Bk_0(1+z)^6\Big)^{\frac{1}{3}}\,. \tag{D4}$$

Note that for $k_0 = 0$ the kinetic term becomes a constant $\widetilde{X} = 3/(8B)$ and the model reduces exactly to ΛCDM. A

$$h(z)^2 = \Omega_{m0}(1+z)^3 - \frac{\widetilde{X}(z)}{3} + \frac{4}{3}B\widetilde{X}(z)^2\,. \tag{D5}$$

Expanding (D5) around future infinity $z \to -1$ reveals the following form of the solution

$$h(z)^2 = \frac{1}{16B} + \left(\Omega_{m0} + \sqrt{\frac{k_0}{6B}}\right)(1+z)^3 + \mathcal{O}(1+z)^6\,. \tag{D6}$$

Appendix

Redshift representation & MCMC setup for $\theta = \{H_0, B, r_d, \mathcal{M}\}$

In terms of redshift, the cosmological equations describing the geometric k-essence model take the following form

$$3h(z)^2 = 3r(z) - \frac{1}{2}\Phi(z)^2 + B\Phi(z)^4\,, \tag{60}$$

$$\frac{dh(z)}{dz} = \frac{3h(z)^2 - \frac{1}{2}\Phi(z)^2 + \frac{1}{3}B\Phi(z)^4}{2(1+z)h(z)}\,, \tag{61}$$

$$\frac{d\Phi(z)}{dz} = \frac{-4Bh(z)\Phi(z)^3 + 3h(z)\Phi(z)}{(1+z)h(z)(1-4B\Phi^2)}\,. \tag{62}$$

$$1 = \Omega_{m0} - \frac{1}{6}\Phi_0^2 + \frac{B}{3}\Phi_0^4\,, \tag{63}$$

$$50 \leq H_0 \leq 100\,, \quad \frac{1}{16-16\Omega_{m0}} \leq B < \frac{4-3\Omega_{m0}}{6(\Omega_{m0}-2)^2}\,, \quad 100 \leq r_d \leq 300\,, \quad -20 \leq \mathcal{M} \leq -18\,, \tag{82}$$